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TWO-WEIGHT NORM INEQUALITIES FOR THE CESÀRO MEANS OF HERMITE EXPANSIONS

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ABSTRACT. An accurate estimate is obtained of the Cesàro kernel for Hermite expansions. This is used to prove two-weight norm inequalities for Cesàro means of Hermite polynomial series and for the supremum of these means. These extend known norm inequalities, even in the single power weight and "unweighted" cases. An almost everywhere convergence result is obtained as a corollary. It is also shown that the conditions used to prove norm boundedness of the means and most of the conditions used to prove the boundedness of the Cesàro supremum of the means are necessary.

1. INTRODUCTION

The purpose of this paper is to extend the results of [8] and [4] concerning Cesàro means of Laguerre expansions to Hermite expansions. The main part is concerned with finding an accurate estimate of the Cesàro-Hermite kernel. This is then used to prove inequalities of the form

$$\sup_{n\geq 0}\Big\||x|^a(1+|x|)^{b-a}\sigma_{\delta,n}(f,x)\Big\|_p \leq C\Big\||x|^A(1+|x|)^{B-A}f(x)\Big\|_p \tag{1.1}$$

and

$$\Big\||x|^a(1+|x|)^{b-a}\sup_{n\geq 0}|\sigma_{\delta,n}(f,x)|\Big\|_p \leq C\Big\||x|^A(1+|x|)^{B-A}f(x)\Big\|_p \tag{1.2}$$

where $\sigma_{\delta,n}(f,x)$ is the nth Cesàro mean of order δ of the expansion of $f(x)$ in orthonormalized Hermite polynomials, $\|g\|_p$ denotes the unweighted L^p norm on $(-\infty,\infty)$ and C is independent of f. An almost everywhere convergence result is also proved.

The kernel estimate is given in Theorem (4.5). It is obtained by writing the kernel in terms of Cesàro-Laguerre kernels and using the estimate in [8]. If the order of summation, δ, is an integer, this is simple since the Cesàro-Hermite kernel is a linear combination of a fixed finite number of Cesàro-Laguerre kernels. This fact was used in [5]. For δ not an integer, obtaining the estimate is more complicated because the expression contains $n+1$ terms; this is given in Lemma (3.8). This is used with the estimate from [8] to obtain the estimate in §4. Because of the similarity of the estimate to the Laguerre case, the norm inequalities, necessity

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results and convergence results proved in §5 are essentially corollaries of the results in [4] for Laguerre expansions with parameter $\alpha = -1/2$.

Because of the generality of the weight functions used in §5, the results are somewhat complicated. We, therefore, state simple consequences here of the theorems in §5 restricted to the single weight x^r. They are as follows.

Theorem (1.3). *If $1 \le p \le \infty$ and $\delta > 0$, then*

$$\sup_{n\ge 0}\left\| |x|^r \sigma_{\delta,n}(f,x)\right\|_p \le C \left\| |x|^r f(x)\right\|_p, \tag{1.4}$$

with C independent of f, if and only if

$$\delta \ge \max\left(\frac{2}{3p}-\frac{1}{2}, \frac{1}{6}-\frac{2}{3p}\right) \tag{1.5}$$

and

$$\begin{cases} -1 < r \le 0 & p = 1, \\ -\frac{1}{p} < r < 1-\frac{1}{p} & 1 < p < \infty, \\ \ \ 0 \le r < 1 & p = \infty. \end{cases} \tag{1.6}$$

Theorem (1.7). *If $1 < p \le \infty$, $\delta \ge 1/6$ and (1.6), then*

$$\left\| |x|^r \sup_{n\ge 0} |\sigma_{\delta,n}(f,x)| \right\|_p \le C \left\| |x|^r f(x)\right\|_p \tag{1.8}$$

with C independent of f.

Note that since (1.8) implies (1.4), Theorem (1.3) shows that the condition (1.6) is in fact a necessary condition for (1.8). The condition $\delta \ge 1/6$ is not necessary; see, for example, Lemma (3.5), page 238 of [2]. A result could be obtained for $\delta < 1/6$ similar to Theorem (1.16) of [4].

Theorem (1.9). *If $\mu > 0$, $\delta \ge 1/6$, $-1 < r < 0$ and E_μ is the set where $|x|^r \sup_{n\ge 0}(|\sigma_{\delta,n}(f,x)|) > \mu$, then $|E_\mu| \le (C/\mu)\left\| |x|^r f(x)\right\|_1$ holds with C independent of f and μ.*

Theorem (1.10). *If $1 \le p \le \infty$, $\delta > 0$,*

$$\max\left(-\frac{1}{p}-2\delta, -1+\frac{1}{3p}-2\delta\right) \le r < 1-\frac{1}{p},$$

with the left inequality strict if $p = 4/3$ and the right inequality weak if $p = 1$, and $\left\| |x|^r f(x)\right\|_p \le \infty$, then $\lim_{n\to\infty} \sigma_{\delta,n}(f,x) = f(x)$ for almost every x.

Previous results concerning the inequalities (1.1) and (1.2) and almost everywhere convergence include the following. Poiani in Theorem 6, page 29 of [5] considered the case $a = A$ and $b = B$ in (1.1) for $\delta = 1$. She obtained the same results as in Theorem (5.1) except for not including the case $a = 0$ when $p = 1$ or $p = \infty$. Thangavelu in Theorem 5.4.1, page 131 of [7] shows that (1.4) with $r = 0$ holds for $1 \le p \le \infty$ provided $\delta > 1/6$. Theorem 1.4 gives (1.4) for $\delta \ge 1/6$. Theorem 5.4.4, page 135 of [7] shows (1.4) for $r = 0$ for $0 < \delta < 1/6$ but requires strict inequality in (1.5). Similarly, Theorem 5.4.2, page 132 of [7] for $r = 0$ gives (1.8) if $1 < p \le \infty$ and weak type if $p = 1$ provided $\delta > 1/6$ while Theorems (1.7) and (1.9) require only $\delta \ge 1/6$. Theorem 5.4.2 of [7] also asserts almost everywhere convergence for $1 \le p \le \infty$ and $\delta > 1/6$ while Theorem (1.10) requires only that $\delta > 0$.

Throughout this paper C will be used for positive constants independent of f, n, x and y but not necessarily the same at every occurrence. The symbol $\approx$ will mean that the expression on the left is bounded above by C times the expression on the right and vice versa. The expression $[w]$ is the greatest integer less than or equal to w.

2. Preliminaries about Laguerre and Hermite series and functions

The sequence of Hermite functions $\{\mathcal{H}_n\}_{n=0}^{\infty}$ is defined by

$$\mathcal{H}_n(x) := \pi^{-\frac{1}{4}} \left(2^n n!\right)^{-\frac{1}{2}} e^{-x^2/2} H_n(x) \qquad x \in \mathbf{R}$$

where each H_n is an nth degree polynomial that is uniquely defined as on page 105 of [6] by

(a) the leading coefficient of the polynomial is 2^n; and
(b) the $\mathcal{H}_n$ are orthonormal.

Given a real number $\alpha > -1$, the sequence of **Laguerre** functions $\left\{\mathcal{L}_n^{(\alpha)}\right\}_{n=0}^{\infty}$ is defined by

$$\mathcal{L}_n^{(\alpha)}(x) := \Gamma(\alpha+1)^{-\frac{1}{2}} \binom{\alpha+n}{n}^{-\frac{1}{2}} x^{\frac{\alpha}{2}} e^{-\frac{x}{2}} L_n^{(\alpha)}(x) \qquad x > 0$$

where each $L_n^{(\alpha)}$ is an nth degree polynomial that is uniquely defined as on page 100 of [6] by

(a) the leading coefficient of the polynomial is $(-1)^n/n!$; and
(b) the $\mathcal{L}_n^{(\alpha)}$ are orthonormal.

By (5.6.1), page 106 of [6], each $\mathcal{H}_n$ can be expressed in terms of some $\mathcal{L}_n^{(\alpha)}$; namely, given a nonnegative integer n and a real number x,

$$\begin{aligned} \mathcal{H}_{2n}(x) &= (-1)^n |x|^{\frac{1}{2}} \mathcal{L}_n^{\left(-\frac{1}{2}\right)}\left(x^2\right), \text{ and} \\ \mathcal{H}_{2n+1}(x) &= (-1)^n \operatorname{sgn}(x) |x|^{\frac{1}{2}} \mathcal{L}_n^{\left(\frac{1}{2}\right)}\left(x^2\right). \end{aligned} \tag{2.1}$$

The Hermite expansion of a function f is

$$\sum_{n=0}^{\infty} \mathcal{H}_n(x) \left(\int_{-\infty}^{\infty} f(y) \mathcal{H}_n(y)\, dy \right)$$

provided the integrals exist, and for $\delta > 0$ the nth (C, δ)-Cesàro mean of this expansion is

$$\sigma_{\delta,n}(f,x) := \binom{\delta+n}{n}^{-1} \sum_{k=0}^{n} \binom{\delta+n-k}{n-k} \mathcal{H}_n(x) \left(\int_{-\infty}^{\infty} f(y) \mathcal{H}_n(y)\, dy \right).$$

It follows that

$$\sigma_{\delta,n}(f,x) = \int_{-\infty}^{\infty} f(y) \mathcal{K}_{\delta,n}(x,y)\, dy, \tag{2.2}$$

where

$$\mathcal{K}_{\delta,n}(x,y) := \binom{\delta+n}{n}^{-1} \sum_{k=0}^{n} \binom{\delta+n-k}{n-k} \mathcal{H}_k(x) \mathcal{H}_k(y). \tag{2.3}$$

Similarly, the nth (C,δ)-Cesàro mean for a Laguerre expansion satisfies

$$\sigma_{\delta,n}^{(\alpha)}(f,x) = \int_0^\infty f(y)\mathcal{K}_{\delta,n}^{(\alpha)}(x,y)\,dy, \tag{2.4}$$

where

$$\mathcal{K}_{\delta,n}^{(\alpha)}(x,y) := \binom{\delta+n}{n}^{-1}\sum_{k=0}^{n}\binom{\delta+n-k}{n-k}\mathcal{L}_k^{(\alpha)}(x)\,\mathcal{L}_k^{(\alpha)}(y). \tag{2.5}$$

3. Cesàro-Hermite kernels in terms of Cesàro-Laguerre kernels

In this section, we express any (C,δ)-Cesàro kernel of Hermite expansions as a finite sum of (C,δ)-Cesàro kernels of Laguerre expansions. The sum is used in §4 to estimate the size of the given (C,δ)-Cesàro kernel of Hermite expansions.

Given a nonnegative integer n, and real numbers x and y, Poiani in [5], p. 28 used some binomial identities to write a given $(C,1)$-Cesàro kernel of Hermite expansions, $\mathcal{K}_{1,2n}(x,y)$ or $\mathcal{K}_{1,2n+1}(x,y)$, as a sum of Laguerre kernels, $\mathcal{K}_{1,k}^{(\alpha)}(x^2,y^2)$, with $k\in\{n-1,n,n+1\}$, and $\alpha\in\left\{\pm\frac{1}{2}\right\}$. She then used her heretofore established estimate of the size of a given $(C,1)$-Cesàro kernel of Laguerre expansions to estimate the size of the given $(C,1)$-Cesàro kernel of Hermite expansions. Following her lead, we shall

1. establish a binomial identity in §3;
2. use the binomial identity to express a given (C,δ)-Cesàro kernel of Hermite expansions as a sum of (C,δ)-Cesàro kernels of Laguerre expansions also in §3;
3. use Webb's heretofore established estimate of the size of a given (C,δ)-Cesàro kernel of Laguerre expansions to estimate the size of the given (C,δ)-Cesàro kernel of Hermite expansions in §4;
4. use the estimate to prove theorems concerning norm inequalities and almost every convergence of (C,δ)-Cesàro sums of Hermite expansions in §5.

Given real α and a nonnegative integer n, the binomial identity

$$\binom{\alpha-1+n}{n} = \sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{\alpha-1+k}{k}\binom{\alpha}{n-2k} \tag{3.1}$$

is used in the derivations of (3.6) and (3.7). The identity is established by use of the identity

$$(1-x)^{-\alpha} = (1-x^2)^{-\alpha}(1+x)^{\alpha}, \tag{3.2}$$

where α is real and $x\neq\pm1$. In particular, each $\binom{\alpha}{n}$ is a coefficient of the Maclaurin series of $(1+x)^\alpha = \sum_{n=0}^{\infty}\binom{\alpha}{n}x^k$; and each coefficient of the Maclaurin series of $(1-x)^{-\alpha} = \sum_{n=0}^{\infty}(-1)^n\binom{-\alpha}{n}x^n$ is equal to $\binom{\alpha-1+n}{n}$. Let $a_n := \binom{\alpha-1+n}{n}$

and $b_n := \binom{\alpha}{n}$. For $|x| < 1$ we have from (3.2) that

$$\sum_{n=0}^{\infty} a_n x^n = \left\{ \sum_{n=0}^{\infty} a_n x^{2n} \right\} \left\{ \sum_{n=0}^{\infty} b_n x^n \right\} = \sum_{n=0}^{\infty} \left\{ x^n \sum_{k=0}^{\left[\frac{n}{2}\right]} a_k b_{n-2k} \right\}. \tag{3.3}$$

Comparing the left and right sides of (3.3) shows that $a_n = \sum_{k=0}^{\left[\frac{n}{2}\right]} a_k b_{n-2k}$; this is (3.1).

Given nonnegative δ, a nonnegative integer n, and real x and y, (2.3) defines $\mathcal{K}_{\delta,n}(x,y)$, a Cesàro kernel of Hermite expansions. Rearranging the sum in (2.3) by separately summing over the even indices, and the odd indices, we see that $\mathcal{K}_{\delta,n}(x,y)$ is the sum of

$$\binom{\delta+n}{n}^{-1} \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{\delta+n-2k}{n-2k} \mathcal{H}_{2k}(x)\, \mathcal{H}_{2k}(y)$$

and

$$\binom{\delta+n}{n}^{-1} \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{\delta+n-2k-1}{n-2k-1} \mathcal{H}_{2k+1}(x)\, \mathcal{H}_{2k+1}(y).$$

Applying Equations (2.1) to replace the Hermite functions by Laguerre functions, we see that $\mathcal{K}_{\delta,n}(x,y)$ is the sum of

$$|xy|^{\frac{1}{2}} \binom{\delta+n}{n}^{-1} \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{\delta+n-2k}{n-2k} \mathcal{L}_k^{\left(-\frac{1}{2}\right)}\left(x^2\right) \mathcal{L}_k^{\left(-\frac{1}{2}\right)}\left(y^2\right) \tag{3.4}$$

and

$$\operatorname{sgn}(xy)|xy|^{\frac{1}{2}} \binom{\delta+n}{n}^{-1} \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{\delta+n-2k-1}{n-2k-1} \mathcal{L}_k^{\left(\frac{1}{2}\right)}\left(x^2\right) \mathcal{L}_k^{\left(\frac{1}{2}\right)}\left(y^2\right). \tag{3.5}$$

Equation (3.1) says that $\binom{\delta+n-2k}{n-2k}$ is equal to

$$\sum_{j=0}^{\left[\frac{n-2k}{2}\right]} \binom{\delta+j}{j} \binom{\delta+1}{n-2k-2j} = \sum_{j=k}^{\left[\frac{n}{2}\right]} \binom{\delta+j-k}{j-k} \binom{\delta+1}{n-2j};$$

thus (3.4) is rewritten as

$$|xy|^{\frac{1}{2}} \binom{\delta+n}{n}^{-1} \sum_{k=0}^{\left[\frac{n}{2}\right]} \left\{ \mathcal{L}_k^{\left(-\frac{1}{2}\right)}\left(x^2\right) \mathcal{L}_k^{\left(-\frac{1}{2}\right)}\left(y^2\right) \sum_{j=k}^{\left[\frac{n}{2}\right]} \binom{\delta+j-k}{j-k} \binom{\delta+1}{n-2j} \right\}.$$

Interchanging the order of summation, and using (2.5), we see that the last expression is equal to

$$|xy|^{\frac{1}{2}} \binom{\delta+n}{n}^{-1} \sum_{j=0}^{\left[\frac{n}{2}\right]} \binom{\delta+1}{n-2j} \binom{\delta+j}{j} \mathcal{K}_{\delta,j}^{\left(-\frac{1}{2}\right)}\left(x^2, y^2\right). \tag{3.6}$$

This expression is equal to (3.4). That (3.5) is equal to

$$\operatorname{sgn}(xy)|xy|^{\frac{1}{2}}\binom{\delta+n}{n}^{-1}\sum_{j=0}^{\left[\frac{n-1}{2}\right]}\binom{\delta+1}{n-1-2j}\binom{\delta+j}{j}\mathcal{K}_{\delta,j}^{\left(\frac{1}{2}\right)}\left(x^2,y^2\right) \tag{3.7}$$

is proved in the same manner, and we leave the details to the reader. This discussion proves the following result.

Lemma (3.8). *Let* $\delta \geq 0$, $n \geq 0$ (*where* n *is an integer*), x *and* y *real be given. Then* $\mathcal{K}_{\delta,n}(x,y)$, *the nth* (C,δ)*-Cesàro kernel of Hermite expansions, is the sum of* (3.6) *and* (3.7).

4. An estimate of the modulus of the Cesàro-Hermite kernel

In this section we establish an estimate for the modulus of the (C,δ)-Cesàro kernel of Hermite expansions. The calculations are rather technical and rely on estimates of the moduli of Laguerre functions established by Askey and Wainger. Basing their work on an asymptotic form of the Laguerre polynomial given by Erdélyi, Askey and Wainger gave their estimates in tabular form. Following Poiani, we shall express the estimates in the form of a function. To this end, we introduce the following notation to capture the Askey and Wainger estimates. Let α, η, n, x and y be real numbers. Define

$$\nu^{(\alpha)}(n) := 4n+2\alpha+2,$$

$$z_n^{(\alpha)}(x,y) := \left|\nu^{(\alpha)}(n)-x\right| + \left|\nu^{(\alpha)}(n)-y\right| + \sqrt[3]{\nu^{(\alpha)}(n)},$$

$$z_n^{(\alpha)}(x) := \left|\nu^{(\alpha)}(n)-x\right| + \sqrt[3]{\nu^{(\alpha)}(n)},$$

$$\Phi_n^{(\alpha)}(x) := \begin{cases} 1 & 0 \leq x \leq \nu^{(\alpha)}(n), \\ \exp\left\{-\eta\sqrt{\dfrac{(x-\nu^{(\alpha)}(n))^3}{\nu^{(\alpha)}(n)}}\right\} & \nu^{(\alpha)}(n) \leq x \leq 2\nu^{(\alpha)}(n), \\ \exp\left\{-\dfrac{\eta x}{2}\right\} & 2\nu^{(\alpha)}(n) \leq x, \end{cases}$$

$$\begin{aligned}
\mathcal{M}_n^{(\alpha)}(x) &:= x^{\frac{\alpha}{2}}\left(\nu^{(\alpha)}(n)^{-1}+x\right)^{-\frac{\alpha}{2}-\frac{1}{4}} z_n^{(\alpha)}(x)^{-\frac{1}{4}}\Phi_n^{(\alpha)}(x), \\
z_n(x) &:= z_{n/2}^{(-1/2)}(x) = |2n+1-x| + \sqrt[3]{2n+1}, \\
z_n(x,y) &:= z_{n/2}^{(-1/2)}(x,y) = |2n+1-x| + |2n+1-y| + \sqrt[3]{2n+1}, \\
\Phi_n(x) &:= \Phi_{n/2}^{(-1/2)}(x), \\
\mathcal{M}_n(x) &:= z_n(x^2)^{-\frac{1}{4}}\Phi_n(x^2) = \mathcal{M}_{n/2}^{(-1/2)}\left(x^2\right)|x|^{\frac{1}{2}}.
\end{aligned}$$

Since $\nu^{\left(\frac{1}{2}\right)}(n) = \nu^{\left(-\frac{1}{2}\right)}\left(n+\frac{1}{2}\right)$, we have $\Phi_n^{(1/2)}(x) = \Phi_{n+1/2}^{(-1/2)}(x)$, and

$$z_n^{(1/2)}(x) = z_{n+1/2}^{(-1/2)}(x); \tag{4.1}$$

it follows easily that

$$\mathcal{M}_n^{(1/2)}\left(x^2\right) \le |x|^{-\frac{1}{2}}\mathcal{M}_{2n+1}(x)\,. \tag{4.2}$$

Askey and Wainger, [1] page 699, said that given $\alpha \ge 0$, there exists some $\eta > 0$ such that $\left|\mathcal{L}_n^{(\alpha)}(x)\right|$ is uniformly dominated by a constant multiple of $\mathcal{M}_n^{(\alpha)}(x)$, with respect to all positive integers n and all $x > 0$. Muckenhoupt in [3] extended this result to all $\alpha > -1$. Also, using (2.1), this Askey-Wainger estimate for $\left|\mathcal{L}_n^{(\alpha)}(x)\right|$, and (4.2), we have $|\mathcal{H}_n(x)| \le C\mathcal{M}_n(x)$, uniformly with respect to all nonnegative integers n and all real numbers x. Webb in [8] said that given $\alpha > -1$, $\delta \ge 0$ and

$$G_{\delta,n}^{(\alpha)}(x,y) := \left(\frac{z_n^{(\alpha)}(x,y)^2}{1+\frac{(x-y)^2}{x+y}z_n^{(\alpha)}(x,y)}\right)^{\frac{1+\delta}{2}} \frac{\mathcal{M}_n^{(\alpha)}(x)\,\mathcal{M}_n^{(\alpha)}(y)}{(n+1)^\delta}, \tag{4.3}$$

there is an $\eta > 0$ such that the (C,δ)-Cesàro kernel of Laguerre expansions, $\mathcal{K}_{\delta,n}^{(\alpha)}(x,y)$, satisfies

$$\left|\mathcal{K}_{\delta,n}^{(\alpha)}(x,y)\right| \le CG_{\delta,n}^{(\alpha)}(x,y) \tag{4.4}$$

for all nonnegative integers n, all $x > 0$, and all $y > 0$, with C independent of n, x and y.

We shall prove:

Theorem (4.5). *Given $\delta \ge 0$, there is an $\eta > 0$ such that $|\mathcal{K}_{\delta,n}(x,y)|$, the modulus of the (C,δ)-Cesàro kernels of Hermite expansions, is uniformly dominated with respect to all nonnegative integers n, and all $x > 0$, and all $y > 0$, by some constant multiple of*

$$G_{\delta,n}(x,y) := \left(\frac{z_n(x^2,y^2)^2}{1+(|x|-|y|)^2 z_n(x^2,y^2)}\right)^{\frac{1+\delta}{2}} \frac{\mathcal{M}_n(x)\,\mathcal{M}_n(y)}{(n+1)^\delta}. \tag{4.6}$$

Before we prove this theorem, we need some more facts:

Lemma (4.7). *If x and y are real and $\big||x|-|y|\big| \le (2n+1)^{-1/6}$ for some $n \ge 0$, then for $0 \le k \le n$ we have $z_k(x^2) \approx z_k(y^2) \approx z_k(x^2,y^2)$ with constants independent of n, k, x and y.*

It is sufficient to show that $z_k(x^2) \le Cz_k(y^2)$ since symmetry will then complete the proof of the first equivalence and the first equivalence implies the second. Since $z_k(x^2) \le z_k(y^2) + \left|x^2-y^2\right|$, it is sufficient to show that

$$\left|x^2-y^2\right| = \left|\left(2\sqrt{(y^2-2k-1)+2k+1}\right)(|x|-|y|) + (|x|-|y|)^2\right| \le Cz_k(y^2).$$

To do this, estimate the square root using the fact that if $b \ge 1$ and $a+b \ge 0$, then $\sqrt{a+b} \le |a| + \sqrt{b}$. This gives

$$\left|x^2-y^2\right| \le 2\left(\left|2k+1-y^2\right| + \sqrt{2k+1}\right)\Big||x|-|y|\Big| + (|x|-|y|)^2\,,$$

and the result follows from the hypothesis and the definition of z_k.

Lemma (4.8). *If n and k are nonnegative, then $\dfrac{z_k(x)}{z_n(x)} \le C(1+|n-k|)$ and $\dfrac{z_k(x,y)}{z_n(x,y)} \le C(1+|n-k|)$ with C independent of x, y, k and n.*

The first inequality follows from the fact that

$$\frac{z_k(x)}{z_n(x)} = 1 + \frac{|x-2k-1| - |x-2n-1| + \sqrt[3]{2k+1} - \sqrt[3]{2n+1}}{z_n(x)},$$

since the numerator of the fraction has absolute value bounded by $3|n-k|$ and the denominator is bounded below by 1. The second inequality is proved similarly.

Proof of Theorem (4.5). Let $\delta \geq 0$ be given, let n be an arbitrary nonnegative integer, and let x and y be arbitrary real numbers. From (4.4), and the definitions of $\mathcal{M}_n$, and $z_n(x,y)$, we have

$$\left|\mathcal{K}_{\delta,n}^{(-1/2)}\left(x^2,y^2\right)\right| \leq C\left(\frac{z_{2n}(x^2,y^2)^2}{1+\frac{(x^2-y^2)^2}{x^2+y^2}z_{2n}(x^2,y^2)}\right)^{\frac{\delta+1}{2}} \frac{\mathcal{M}_{2n}\left(x\right)\mathcal{M}_{2n}\left(y\right)}{(n+1)^{\delta}\sqrt{|xy|}}.$$

Then, using the fact that

$$\frac{\left(x^2-y^2\right)^2}{x^2+y^2} \geq \left(|x|-|y|\right)^2,$$

and the notation of (4.6), we have

$$\left|\mathcal{K}_{\delta,n}^{(-1/2)}\left(x^2,y^2\right)\right| \leq C|xy|^{-\frac{1}{2}}G_{\delta,2n}(x,y). \tag{4.9}$$

Similarly, using (4.1) and (4.2), we also have

$$\left|\mathcal{K}_{\delta,n}^{(1/2)}\left(x^2,y^2\right)\right| \leq C|xy|^{-\frac{1}{2}}G_{\delta,2n+1}(x,y). \tag{4.10}$$

Using (4.9) and (4.10), with Lemma (3.8), shows that $|\mathcal{K}_{\delta,n}\left(x,y\right)|$ is bounded by $C\binom{\delta+n}{n}^{-1}$ times

$$\sum_{j=0}^{\left[\frac{n}{2}\right]}\binom{\delta+1}{n-2j}\binom{\delta+j}{j}G_{\delta,2j}(x,y) + \sum_{j=0}^{\left[\frac{n-1}{2}\right]}\binom{\delta+1}{n-2j-1}\binom{\delta+j}{j}G_{\delta,2j+1}(x,y).$$

Given a real number δ, it is well known that Stirling's formula implies $\left|\binom{\delta+n}{n}\right| \approx C(n+1)^{\delta}$ for all nonnegative n. Therefore, the identity $\binom{\delta}{n} = (-1)^n\binom{n-\delta-1}{n}$ implies $\left|\binom{\delta}{n}\right| \leq C(n+1)^{-1-\delta}$ for all nonnegative n. Using these facts, changing $2j$ in the first sum and $2j+1$ in the second sum to k, and combining the sums, we see that $|\mathcal{K}_{\delta,n}\left(x,y\right)|$ is uniformly bounded, on the domain of n, x, and y, by

$$C\sum_{k=0}^{n}\left(\frac{z_k\left(x^2,y^2\right)^2}{1+\left(|x|-|y|\right)^2 z_k\left(x^2,y^2\right)}\right)^{\frac{\delta+1}{2}} \frac{(n+1)^{-\delta}\Phi_k\left(x^2\right)\Phi_k\left(y^2\right)}{(n-k+1)^{2+\delta}(z_k(x^2)z_k(y^2))^{\frac{1}{4}}}.$$

Since Φ_k is an increasing function of k, it follows that $|\mathcal{K}_{\delta,n}\left(x,y\right)|$ is uniformly bounded, with respect to n, x, and y, by some constant multiple of the product of

$$\left(\frac{z_n\left(x^2,y^2\right)^2}{1+\left(|x|-|y|\right)^2 z_n\left(x^2,y^2\right)}\right)^{\frac{\delta+1}{2}} \frac{\Phi_n\left(x^2\right)\Phi_n\left(y^2\right)}{(n+1)^{\delta}\left(z_n\left(x^2\right)z_n\left(y^2\right)\right)^{\frac{1}{4}}} \tag{4.11}$$

with

$$\sum_{k=0}^{n} \frac{\left(\frac{z_n\left(x^2\right) z_n\left(y^2\right)}{z_k\left(x^2\right) z_k\left(y^2\right)}\right)^{\frac{1}{4}}\left(\frac{z_k\left(x^2,y^2\right)}{z_n\left(x^2,y^2\right)}\right)^{1+\delta}\left(\frac{1+(|x|-|y|)^2 z_n\left(x^2,y^2\right)}{1+(|x|-|y|)^2 z_k\left(x^2,y^2\right)}\right)^{\frac{1+\delta}{2}}}{(n-k+1)^{2+\delta}}. \tag{4.12}$$

Since (4.11) equals $G_{\delta,n}(x,y)$, it remains to show that (4.12) is uniformly bounded with respect to all nonnegative integers n, and all real numbers x and y.

In case $(|x|-|y|)^2 z_n\left(x^2,y^2\right) \le 1$, we can replace the last factor of each term in (4.12)by $2^{(1+\delta)/2}$; and by Lemma (4.7), we get the bound

$$C\sum_{k=0}^{n}\left(\frac{z_k(x^2)}{z_n(x^2)}\right)^{\frac{1}{2}+\delta}(1+n-k)^{-2-\delta}.$$

Lemma (4.8) then gives the bound $C\sum_{k=0}^{n}(1+n-k)^{-3/2}$ and completes this case.

In case $(|x|-|y|)^2 z_n\left(x^2,y^2\right) > 1$, the last factor of each term in (4.12) can be replaced by $C\left(\frac{z_n(x^2,y^2)}{z_k(x^2,y^2)}\right)^{\frac{1+\delta}{2}}$ to give the bound

$$C\sum_{k=0}^{n}(n-k+1)^{-2-\delta}\left(\frac{z_n\left(x^2\right) z_n\left(y^2\right)}{z_k\left(x^2\right) z_k\left(y^2\right)}\right)^{\frac{1}{4}}\left(\frac{z_k\left(x^2,y^2\right)}{z_n\left(x^2,y^2\right)}\right)^{\frac{1+\delta}{2}}. \tag{4.13}$$

If $z_k(x^2) \ge z_k(y^2)$, then $z_k(x^2) \ge \frac{1}{2} z_k(x^2,y^2)$ and

$$\frac{z_n(x^2)}{z_k(x^2)} \le C\frac{z_n(x^2,y^2)}{z_k(x^2,y^2)}.$$

Using this and Lemma (4.8) then gives

$$\frac{z_n\left(x^2\right) z_n\left(y^2\right)}{z_k\left(x^2\right) z_k\left(y^2\right)} \le C(n-k+1)\frac{z_n(x^2,y^2)}{z_k(x^2,y^2)}. \tag{4.14}$$

By symmetry, (4.14) also holds if $z_k(x^2) \le z_k(y^2)$. We can then use (4.14) in (4.13) to get the estimate

$$\sum_{k=0}^{n}(1+n-k)^{-\frac{7}{4}-\delta}\left(\frac{z_k(x^2,y^2)}{z_n(x^2,y^2)}\right)^{\frac{1}{4}+\frac{\delta}{2}}.$$

Now using Lemma (4.8) again gives the bound $C\sum_{k=0}^{n}(1+n-k)^{-\frac{3}{2}-\frac{\delta}{2}}$. This completes the proof of Theorem (4.5).

5. Norm inequalities and convergence results

Due to the similarity of the estimate for the Cesàro-Hermite kernel and the Cesàro-Laguerre kernel for $\alpha = -1/2$, norm inequalities for Cesàro means of Hermite expansions are easily obtained using the results and methods of [4]. A few typical results and an outline of their proofs are given below. Analogs of other results in [4] could be obtained similarly and are left to the interested reader.

Theorem (5.1). *If $1 \le p \le \infty$ and $\delta > 0$, then*

$$\sup_{n\ge 0} \left\| |x|^a (1+|x|)^{b-a} \sigma_{\delta,n}(f,x) \right\|_p \le C \left\| |x|^A (1+|x|)^{B-A} f(x) \right\|_p \tag{5.2}$$

with C independent of f if and only if

$$a > -\frac{1}{p} \qquad (\ge \text{ if } p=\infty), \tag{5.3}$$

$$A - a \le 0, \tag{5.4}$$

$$A < 1 - \frac{1}{p} \qquad (\le \text{ if } p=1), \tag{5.5}$$

$$b \le 1 - \frac{1}{p} + 2\delta, \tag{5.6}$$

$$b \le \frac{2}{3} + \frac{1}{3p} + 2\delta, \tag{5.7}$$

$$B \ge -\frac{1}{p} - 2\delta, \tag{5.8}$$

$$B \ge -1 + \frac{1}{3p} - 2\delta, \tag{5.9}$$

$$b - B \le 1 - \frac{4}{3p} + 2\delta, \tag{5.10}$$

$$b - B \le 0, \tag{5.11}$$

and

$$b - B \le -\frac{1}{3} + \frac{4}{3p} + 2\delta, \tag{5.12}$$

and in at least one of each of the following pairs the inequality is strict: (5.6) *and* (5.7), (5.7) *and* (5.12), (5.8) *and* (5.9), (5.9) *and* (5.10).

Theorem (5.13). *If $1 < p \le \infty$, $\delta > 0$,* (5.3), (5.4), (5.5), (5.8), (5.9), (5.11),

$$b < \frac{2}{3} - \frac{1}{p} + 2\delta \qquad (\le \text{ if } p=\infty), \tag{5.14}$$

and

$$b - B \le 2\delta - \frac{1}{3}, \tag{5.15}$$

with inequality strict in at least one of (5.8) *and* (5.9) *and at least one of* (5.14) *and* (5.15)*, then*

$$\left\| |x|^a (1+|x|)^{b-a} \sup_{n\ge 0} |\sigma_{\delta,n}(f,x)| \right\|_p \le C \left\| |x|^A (1+|x|)^{B-A} f(x) \right\|_p \tag{5.16}$$

with C independent of f.

As noted after Theorem (1.7), since (5.16) implies (5.2), Theorem (5.1) shows that the conditions in Theorem (5.13) except for (5.14), (5.15) and their pair condition are also necessary for (5.16). Following the method of §10 of [4] a result could be proved with $b - B > 2\delta - 1/3$.

Theorem (5.17). *If $a > -1$, $A \le 0$, $b < 2\delta - 1/3$, $B \ge -2\delta - 2/3$ ($>$ if $b = -1$), $A \le a$ ($<$ if $a = 0$), $b \le B$, $b \le B + 2\delta - 1/3$ and E_μ is the set where $|x|^a(1+|x|)^{b-a}\sup_{n\ge 0}(|\sigma_n(f,x)|) > \mu$, then $|E_\mu| \le (C/\mu)\left\||x|^A(1+|x|)^{B-A}f(x)\right\|_1$ holds with C independent of f.*

Theorem (5.18). *If $1 \le p \le \infty$, $\delta > 0$, (5.5), (5.8) and (5.9) hold with strict inequality in at least one of (5.8) and (5.9) and $\left\||x|^A(1+|x|)^{B-A}f(x)\right\|_p < \infty$, then $\lim_{n\to\infty}\sigma_{\delta,n}(f,x) = f(x)$ for almost every x.*

Theorem (5.18) is proved by choosing an a large enough and a b small enough that a, A, b and B satisfy the conditions of Theorem (5.13) if $p > 1$ or Theorem (5.17) if $p = 1$. The conclusions of those theorems then imply the almost everywhere convergence by a standard argument. Note that Theorem (1.10) does not follow from Theorem (1.7); this illustrates the value of proving more general weight function results like Theorem (5.13) even when only power function results are the ultimate objective.

To prove the sufficiency of the conditions in Theorem (5.1), it is enough by use of Theorem (4.5) to show that (5.2) holds with $\sigma_{\delta,n}(f,x)$ replaced by $\int_{-\infty}^{\infty} G_{\delta,n}(x,y)|f(y)|dy$, where $G_{\delta,n}(x,y)$ is as defined in (4.6). Because this estimate and the weight functions are even in both x and y, it is sufficient to prove the inequality with the integral and the norms restricted to the interval $[0,\infty)$. Then with the change of variables $x = \sqrt{u}$, $y = \sqrt{v}$, it is sufficient to show that

$$\begin{aligned}\left\|u^{a/2-1/(2p)}(1+\sqrt{u})^{b-a}\int_0^\infty G_{\delta,n}(\sqrt{u},\sqrt{v})v^{-1/2}|f(\sqrt{v})|dv\right\|_p \\ \le C\left\|u^{A/2-1/(2p)}(1+\sqrt{u})^{B-A}f(\sqrt{u})\right\|_p,\end{aligned}$$

where the norms are now in the variable u over $[0,\infty)$. Next let $|f(\sqrt{v})| = v^{1/4}g(v)$ and use the facts that $(1+\sqrt{u}) \approx (1+u)^{1/2}$ and $(\sqrt{u}-\sqrt{v})^2 \approx \dfrac{(u-v)^2}{u+v}$. With these changes we see that it is enough to show that

$$\begin{aligned}\left\|\frac{u^{\frac{1}{4}-\frac{1}{2p}+\frac{a}{2}}}{(1+u)^{\frac{a-b}{2}}}\int_0^\infty\left(\frac{(u+v)z_n(u,v)^2}{u+v+(u-v)^2z_n(u,v)}\right)^{\frac{1+\delta}{2}}\frac{\mathcal{M}_n(\sqrt{u})\mathcal{M}_n(\sqrt{v})}{(uv)^{\frac{1}{4}}(n+1)^\delta}g(v)dv\right\|_p \\ \le C\left\|u^{A/2+1/4-1/(2p)}(1+u)^{(B-A)/2}g(u)\right\|_p.\end{aligned}$$

The kernel equals the function $G^{(-1/2)}_{\delta,n/2}(u,v)$ as defined in (4.3) and in (2.4), p. 1124 of [4] with $\xi = \eta/2$ and $\lambda = 1$. The proof of the sufficiency part of Theorem (2.29) of [4] consists of proving this inequality for appropriate values of the exponents of the weights. As shown in that proof, the inequality holds provided that the parameters

$$(5.19)\quad \left(\frac{a}{2}+\frac{1}{4}-\frac{1}{2p},\quad \frac{b}{2}+\frac{1}{4}-\frac{1}{2p},\quad \frac{A}{2}+\frac{1}{4}-\frac{1}{2p},\quad \frac{B}{2}+\frac{1}{4}-\frac{1}{2p},\quad -\frac{1}{2},\quad \delta\right)$$

satisfy the N_p conditions defined on pages 1125–1126 of [4]. It is now a routine procedure to show that these parameters satisfy the N_p conditions if (a,b,A,B)

satisfy the hypotheses of Theorem (5.1). This completes the sufficiency part of Theorem (5.1).

Theorems (5.13) and (5.17) are proved in the same way as the sufficiency proof of Theorem (5.1). The same change of variables will reduce the proof to the sufficiency part of Theorems (2.30) and (2.31) respectively of [4] and it is simple to show that the resulting parameters (5.19) satisfy the S_p conditions.

The necessity of the conditions in Theorem (5.1) is done in the same way as the necessity proofs in §7 of [4]. First, Theorem (5.18) implies for a fixed $\delta > 0$ and $r > 0$ that $|\sigma_{\delta,n}(\chi_{[r,2r]}, x)|$ converges almost everywhere to $\chi_{[r,2r]}(x)$. From Fatou's lemma and (5.2) it follows that

$$\left\|x^a(1+x)^{b-a}\chi_{[r,2r]}(x)\right\|_p \le C\left\|x^A(1+x)^{B-A}\chi_{[r,2r]}(x)\right\|_p ;$$

(5.4) and (5.11) follow from this. Next, a standard argument as given on page 1141 of [4] or page 113 of [7] shows that (5.2) implies

$$\left\||x|^{-A}(1+|x|)^{A-B}\mathcal{H}_n(x)\right\|_{p'}\left\||x|^a(1+|x|)^{b-a}\mathcal{H}_n\right\|_p \le C(n+1)^\delta.$$

The necessity of the rest of the conditions in Theorem (5.1) follows from this and the following lemma.

Lemma (5.20). *If $1 \le p \le \infty$ and $n \ge 2$, then*

$$\left\||x|^a(1+|x|)^{(b-a)}\mathcal{H}_{2n}\right\|_p \ge C\left(n^{-1/4} + n^{b/2-1/4+1/2p} + n^{b/2-1/12-1/(6p)}\right).$$

In addition,

$$\left\||x|^a(1+|x|)^{(-a-1/p)}\mathcal{H}_{2n}\right\|_p \ge Cn^{-1/4}(\log n)^{1/p},$$

$$\left\||x|^a(1+|x|)^{b-a}\mathcal{H}_{2n}(x)\right\|_4 \ge Cn^{b/2-1/8}(\log n)^{1/4}$$

and $\left\||x|^a(1+|x|)^{b-a}\mathcal{H}_n(x)\right\|_p = \infty$ *if* $a \le -1/p$ *and* $p < \infty$ *or* $a < 0$ *and* $p = \infty$.

To prove Lemma (5.20) use the fact that $\mathcal{H}_{2n}(x) = (-1)^n\sqrt{|x|}\mathcal{L}_n^{(-1/2)}(x^2)$ and make a change of variables to show that

$$\left\||x|^a(1+|x|)^{b-a}\mathcal{H}_{2n}\right\|_p \ge C\left\|x^{a/2+1/4-1/(2p)}(1+x)^{(b-a)/2}\mathcal{L}_n^{-1/2}(x)\right\|_p,$$

where the norm on the right is taken over $[0,\infty)$. The result then follows immediately from Lemma (7.2) on page 1142 of [4]. It can also be proved directly using the asymptotic formula (1.5.1) on page 26 of [7] and the fact that Hermite polynomials of even degree have nonzero constant term.

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